DIMACS

Center for Discrete Mathematics & Theoretical Computer Science
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Catenary or parabola, who will tell?

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Once upon a time in Blumenau, Brazil…

→ talks about math, design and Gaudí:

(Amadeo:)
I had two joint talks in Math & Design conference, telling about complexity in design and about the projection of Gaudí into the XXI century

At a given moment of my first talk, the following slide appeared …

Antonio Gaudí “La línea recta es del hombre, la curva pertenece a Dios”

Etapa 2 arquitectura

About the arches appearing on that previous slide, I accidentally commented:

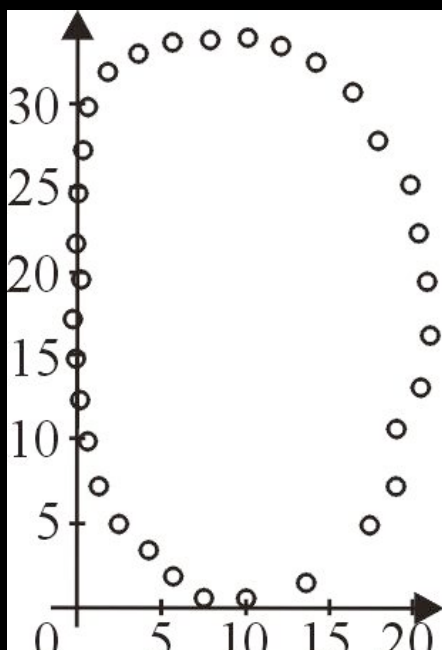
- “Many texts refer to those arches as to be parabolic or catenaries or even others with an absolute lack of rigor, as if those concepts were synonyms and merely colloquial labels”.

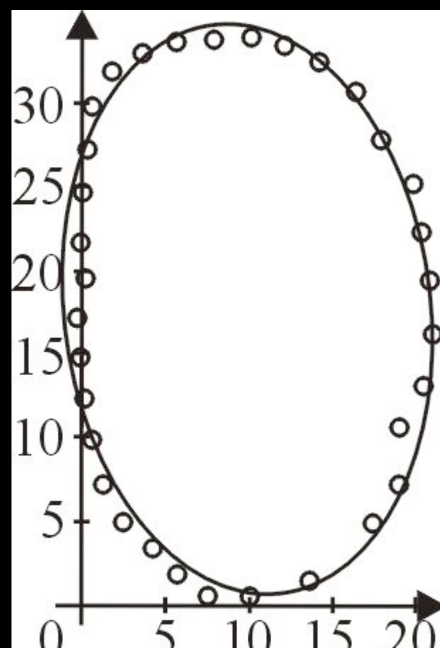
- “In the case of the corridor of the Teresianes’ convent, I have verified that the arches really are parabolic ones”.
- “But in the case of the Palau Güell gates, I have tried several arches and no one have fitted with an acceptable accuracy. So I have to confess that I have no idea about which kind of arch are them, but, at least, I don’t invent an answer where I don’t have any”.

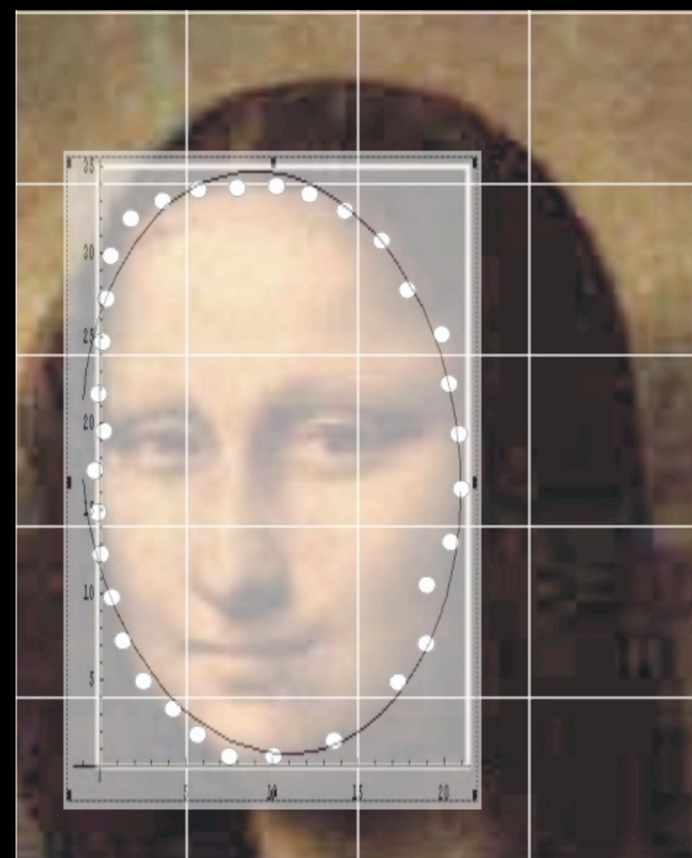
There were questions & someone could not stop

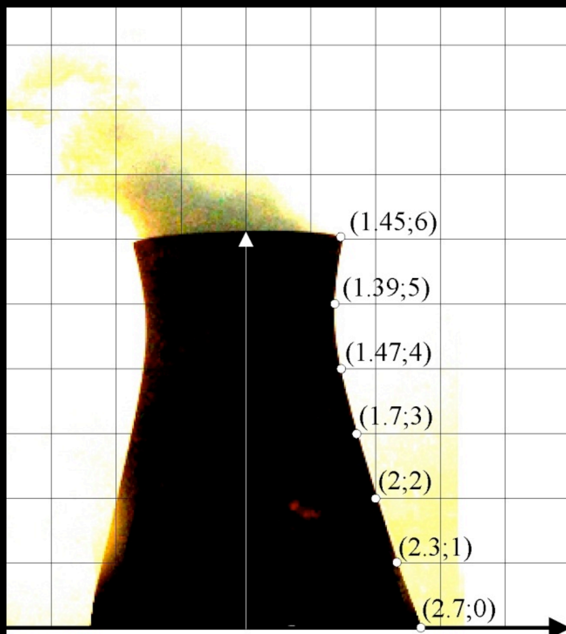
How should “curve fitting” be done?

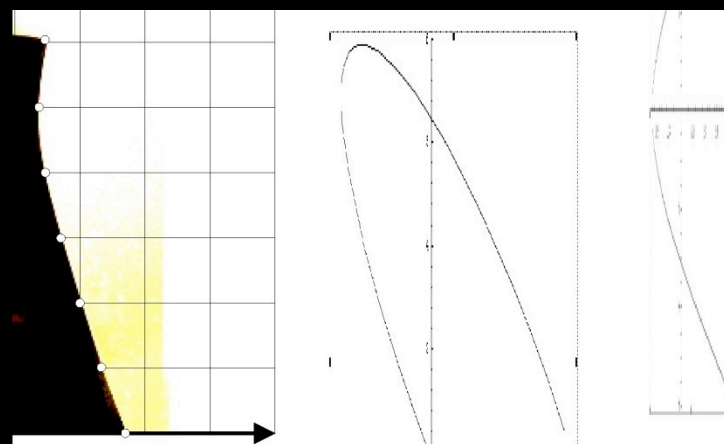
In celestial mechanics, curve fitting procedures are well-known: least-squares, etc.







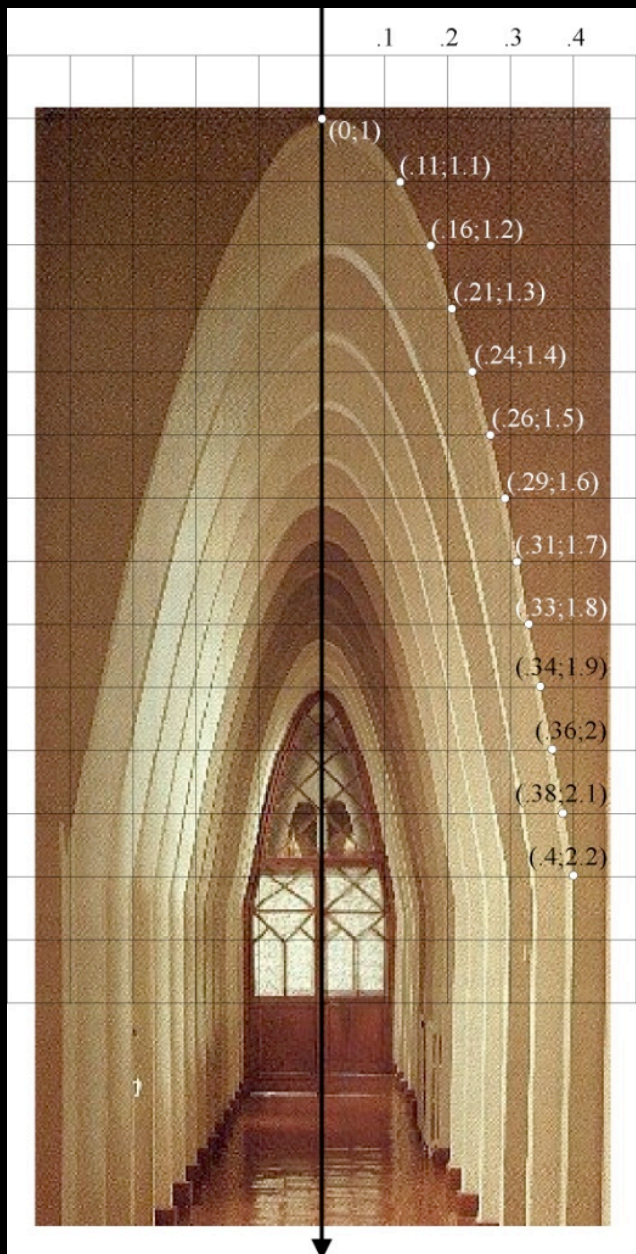
- Note Kepler had an ellipse that almost was a circle, and yet concluded: orbit = ellipse
- 



Nuclear plant: ellipse, not hyperbola (as confirmed by an engineer, afterwards)

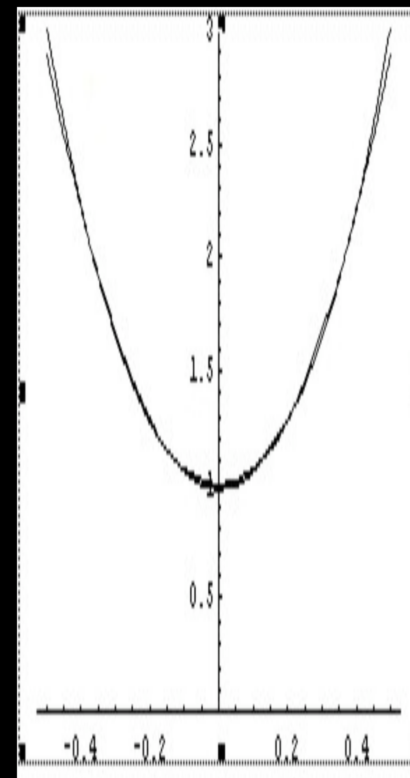
- Paper in Nexus journal (Kim Williams): "Curve fitting in Architecture" (Spring 2007)

- Examples from Gaudí…



Teresianes' Convent

Here, the curving fitting result (Nexus Spring 2007) was confirmed by Amadeo Monreal (Math & Design, June 2007):



Hyperbolic cosine:
$y = -0.7468 + 1.75\cosh(2.8x)$,
99.988% fit.

Parabola:
$y = 0.985 + 7.63x^2$, at 99.985%.

- Examples from Gaudí…

Palau Güell

Hyperbolic cosine:
$y = 1.34 - 0.36\cosh(9.7x)$
fits at 99.88%

Closest parabola:
$y = 1.84 - 52.12x^2$,
fits at only 96.75%

→ The hyperbolic cosine is better!

it is! it is not! it is!

Fortunately, this happened in Brazil…

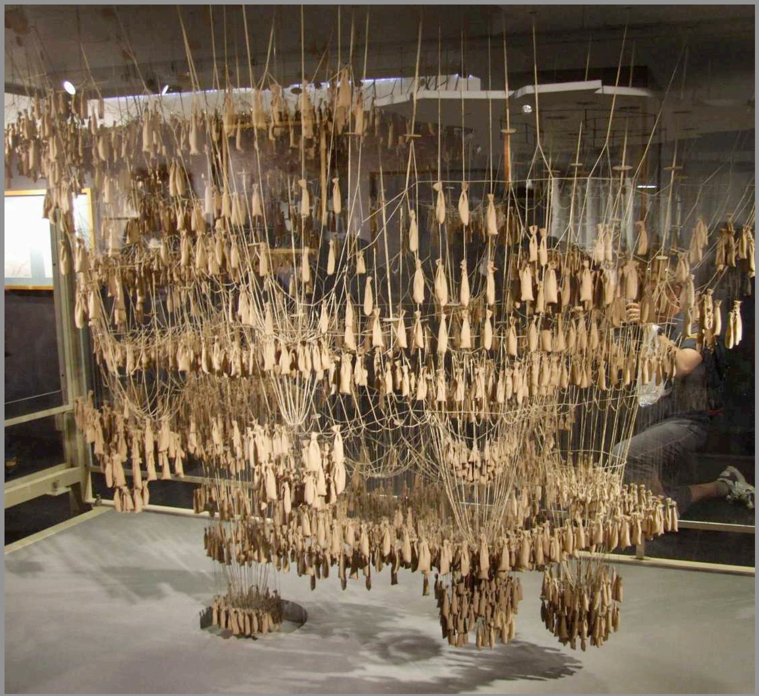
Hyperbolic cosine combination:

$y = c + b \cdot \cosh(x/a)$

Catenary:

$y = c + a \cdot \cosh(x/a)$

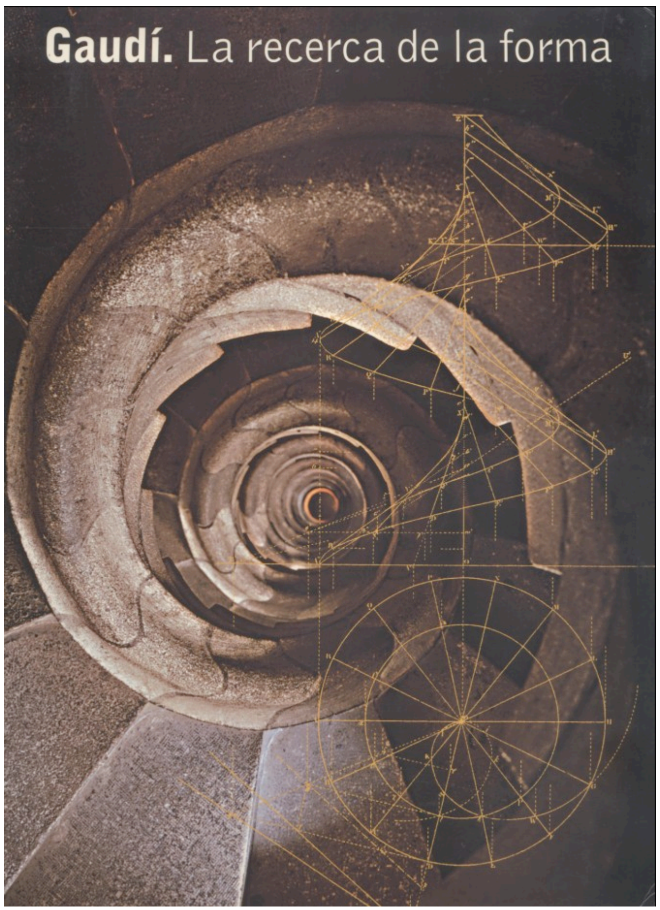
- The "stretched catenary" $y = c + k \cdot a \cdot \cosh(x/a)$

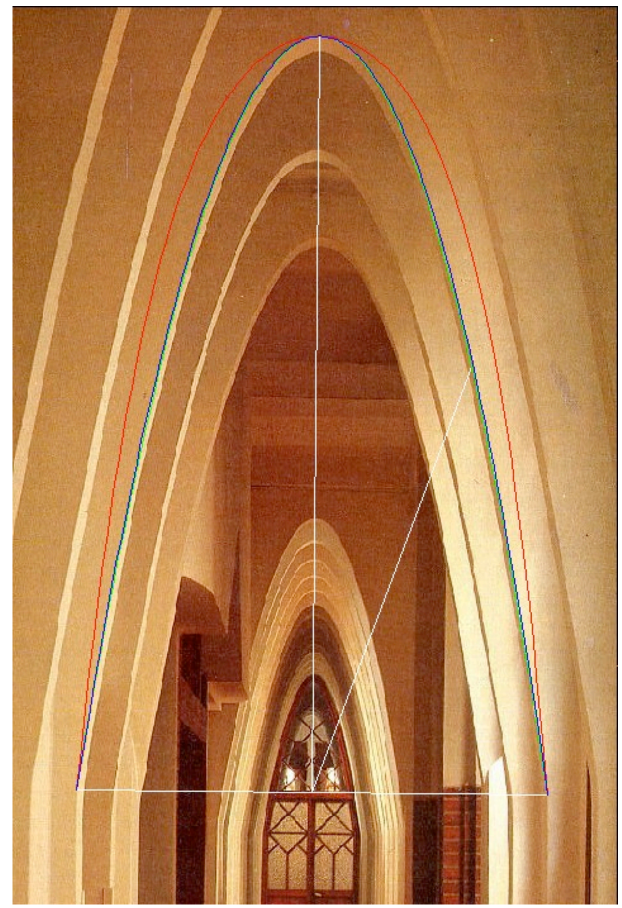


Difference between an idea and the actual result

Back in Spain…

- I kept wondering about Gaudí’s curves – after all, I live in Barcelona!
- Would there be an easy method to settle the question?

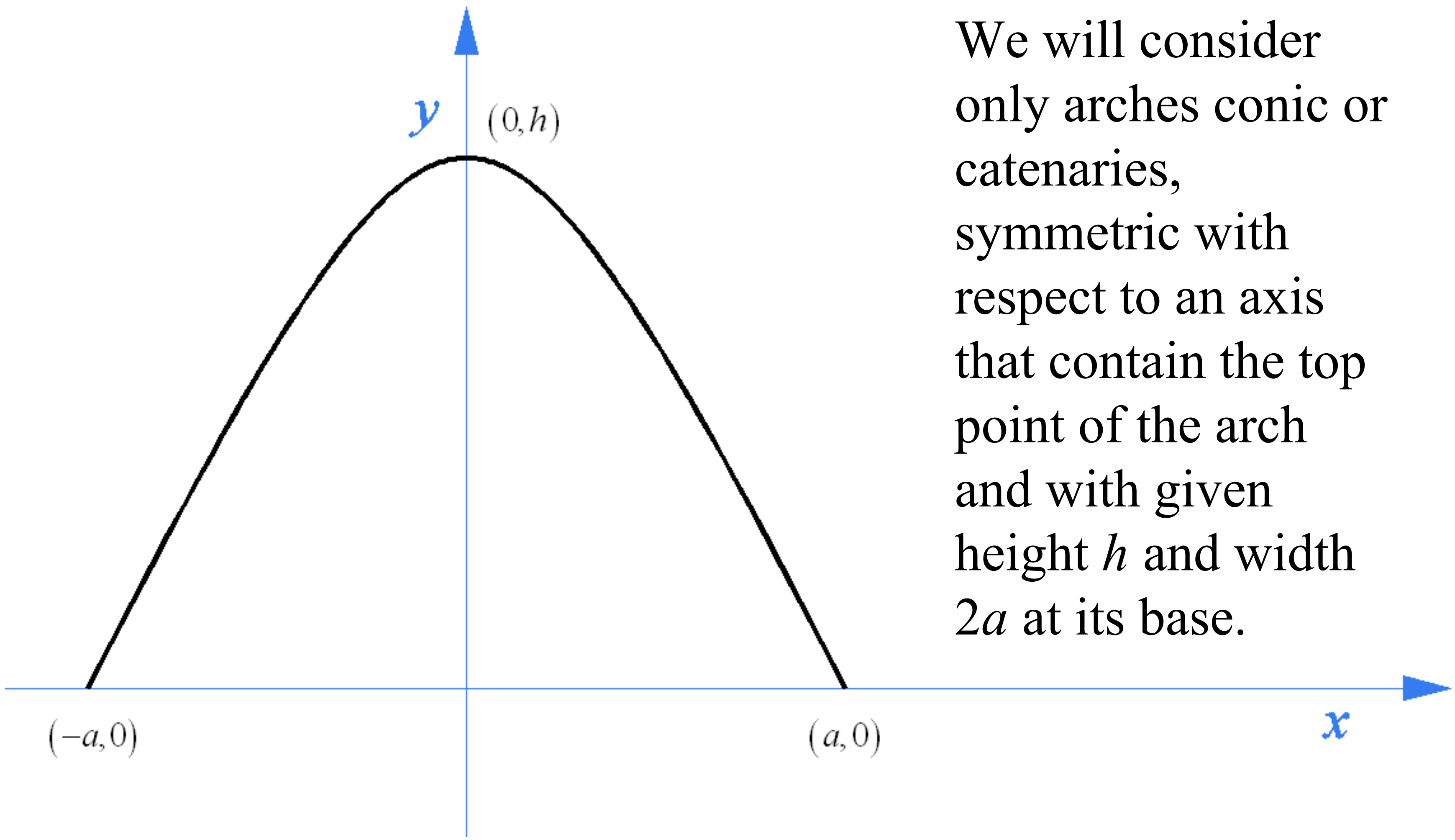




Previous assumptions

- When building houses, the precision from the architect’s plan until the carrying out by a bricklayer is lower than in other technical trades. So, if visually two arches cannot be distinguished, that is enough to accept that both arches coincide.
- Due to the surrounding constrains, the width and the height of an arch are established before to decide its shape.

According to that and conform to Gaudí's style



We will consider only arches conic or catenaries, symmetric with respect to an axis that contain the top point of the arch and with given height h and width $2a$ at its base.

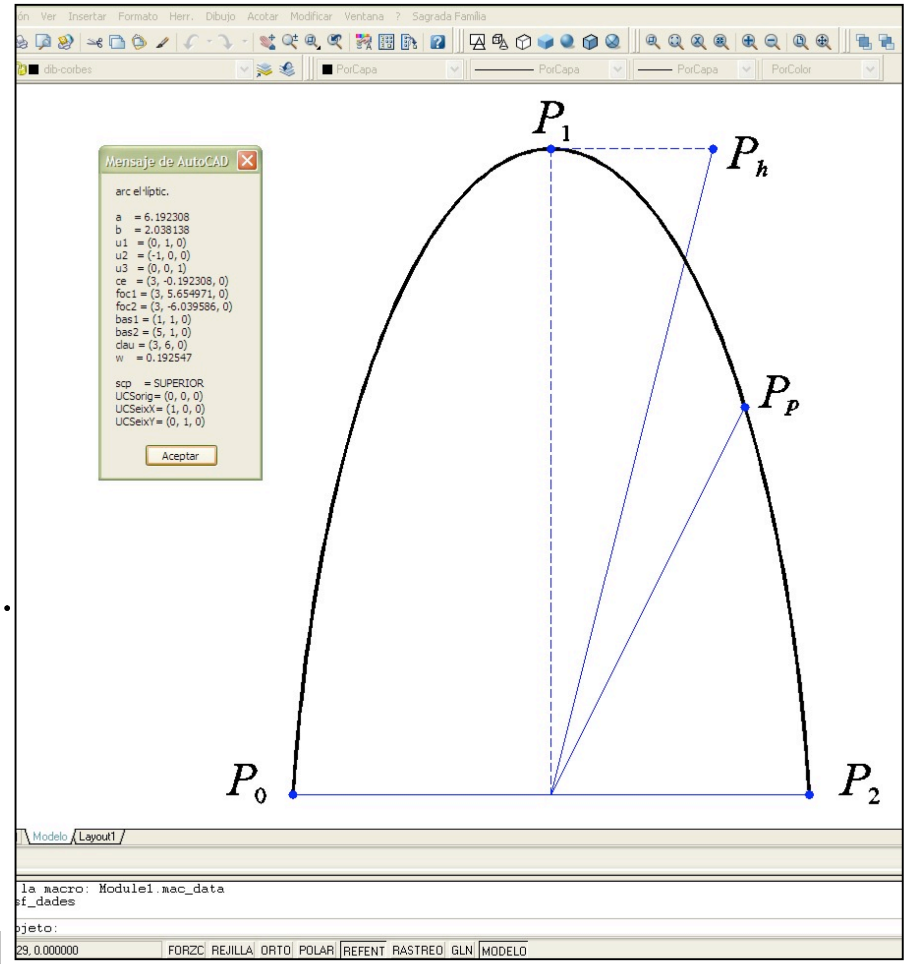
Such ‘straightforward’ method would provide:

1. **An application** that, given some set of points as data input, generate an arch of one of the following types:
   - Catenary
   - Parabolic
   - Circular
   - Conic of any kind (+ information about kind of conic)

2. **A procedure** to confront one of the previous arches with the actual one displayed on a photography

The application

1. Choose the type of arch:
   - Catenary
   - Parabolic
   - Circular
   - Conic of any kind
2. Provide the determining points:
   - First point of the basis of the arch, P_0.
   - Second point of the basis of the arch, P_2.
   - A point to determine the plain of the arch, P_P.
   - A point, P_h, to determine, by projection, the top point of the arch, P_1.
   - For the general conic case, an additional data is needed: either another passing point (it can be just the previous P_P) or a real coefficient w greater than -1.



The points can be either 2D or 3D and placed in any position.

When we said "For the general conic case, an additional data is needed, either another point or a coefficient *w*", this *w* indicated the following:

$$\begin{cases} -1 < w < 1 & \Rightarrow \quad \text{ellipse (possibly cercle)} \\ w = 1 & \Rightarrow \quad \text{parabola} \\ 1 < w & \Rightarrow \quad \text{hyperbola} \end{cases}$$

$$\begin{cases} \lim w \to -1 & \Rightarrow \quad \text{two parallel lines} \\ \lim w \to \infty & \Rightarrow \quad \text{polyline } P_0 - P_1 - P_2 \end{cases}$$

If the user provides a passing point, the program calculates the coefficient w.

$$\begin{cases} -1 < w < 1 & \Rightarrow \quad \text{ellipse (possibly cercle)} \\ w = 1 & \Rightarrow \quad \text{parabola} \\ 1 < w & \Rightarrow \quad \text{hyperbola} \end{cases}$$

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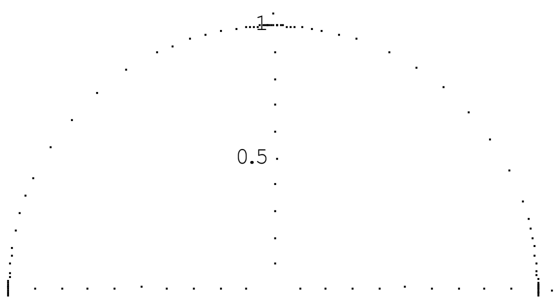
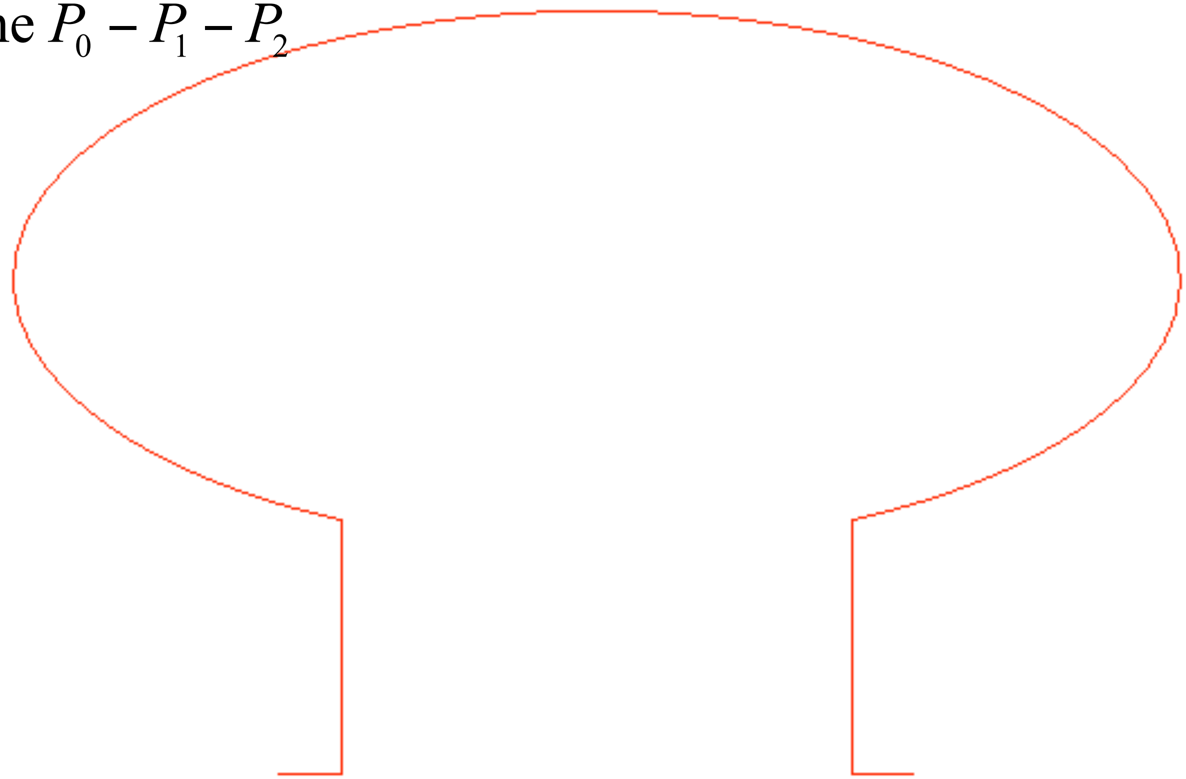
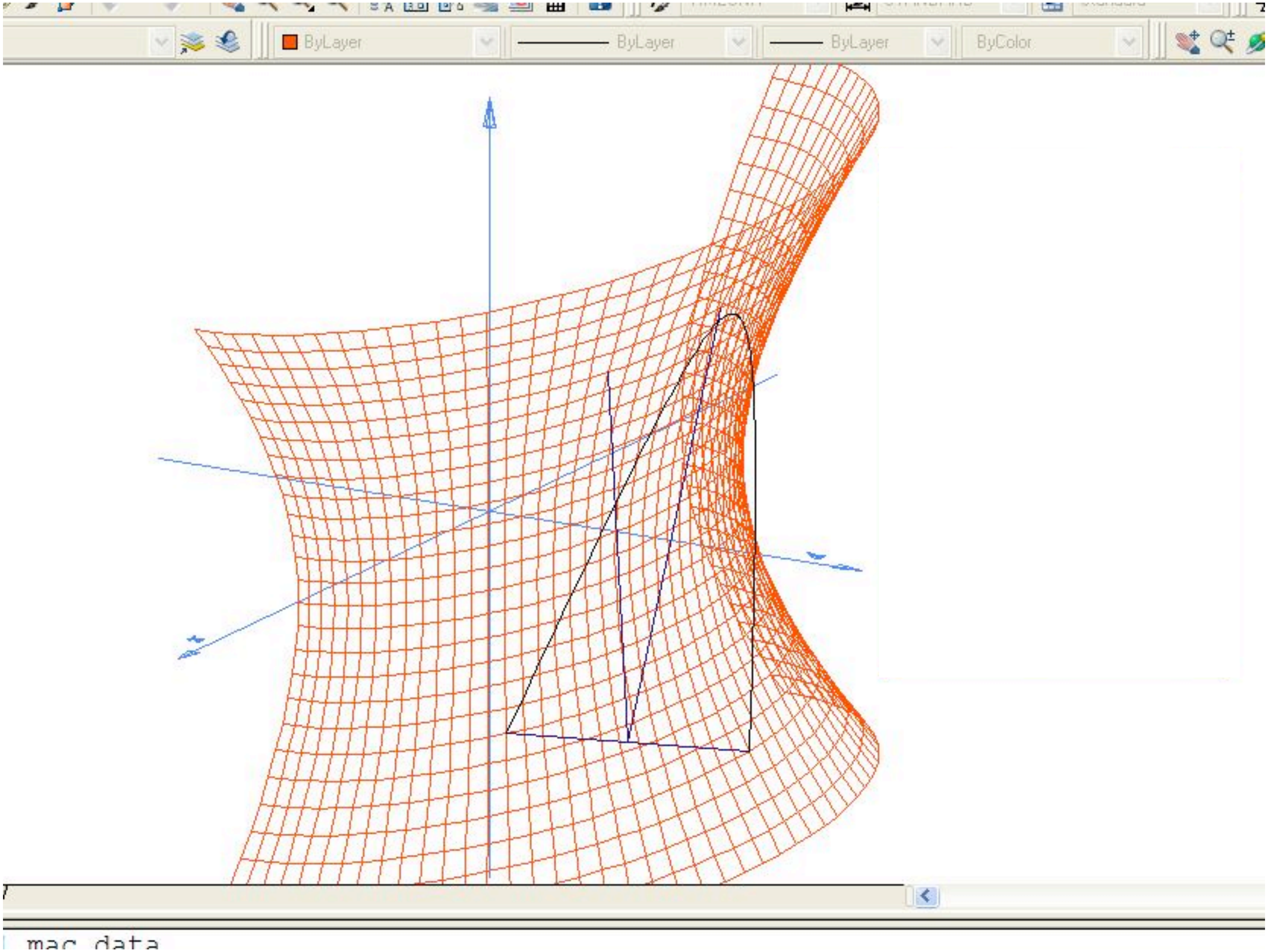




$$\begin{cases} x' = \dfrac{x}{1 + w \cdot y} \\ y' = \dfrac{(1 + w) \cdot y}{1 + w \cdot y} \end{cases}$$



The user can ask the program some information about the arch

The procedure

1. Insert a picture with a front view of the actual arch to be cheeked into the selected CAD program.
2. Determine visually the top point P_1.
3. Trace a circle with centre in P_1 to obtain two symmetric basis points P_0 and P_2.
4. Trace the line P_0 - P_2 and the axis from the middle point of this line to P_1.
5. Trace a line from that middle point to an arbitrary passing point P_P chosen visually.

Now, the user can try to fit any of the arches of the previous application to the actual arch, using the endpoints of the above lines.

Examples

Color code:

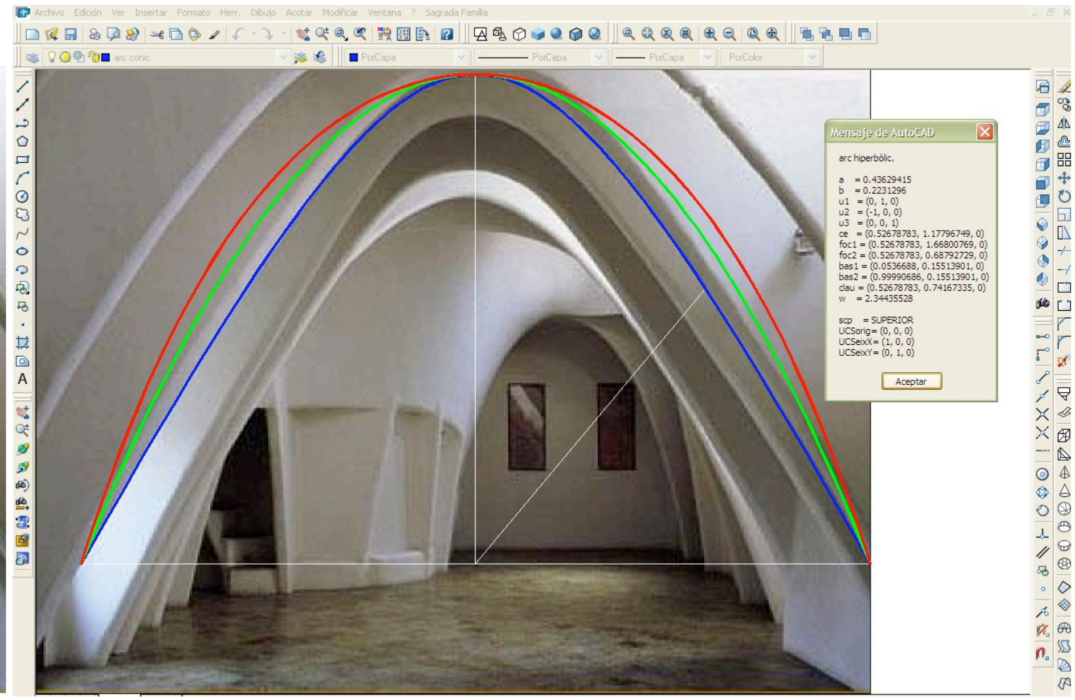
- **catenary**
- **parabola**

The arches of the corridor of the Teresianes’ convent are parabolas

Examples

Color code:

- **catenary**
- **parabola**
- **conic**

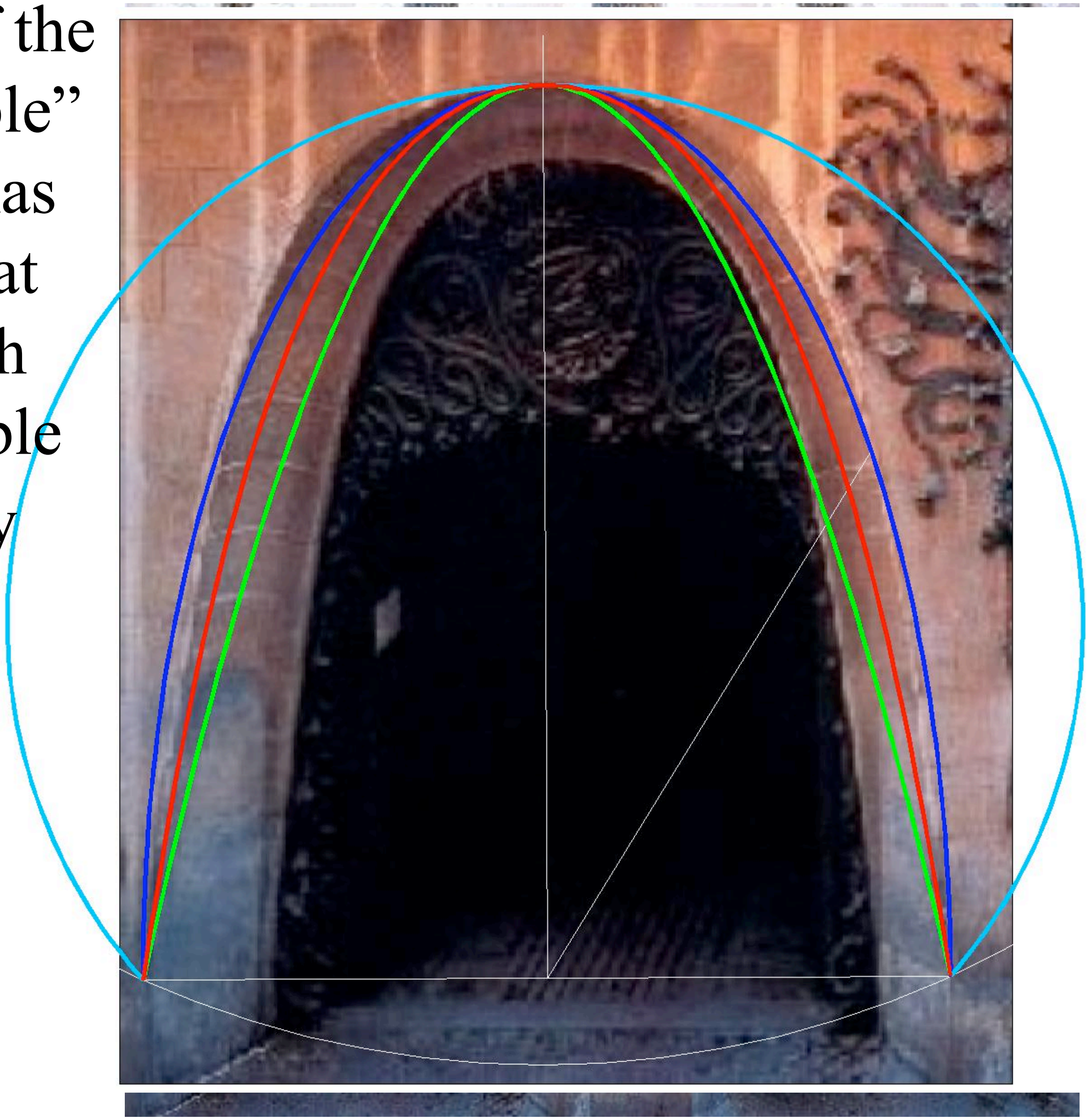


These arches of the attic of the Batlló house are hyperbolas ($w = 2.34$)

But, now, let us turn to …

A special case: the Palau Güell's Gates

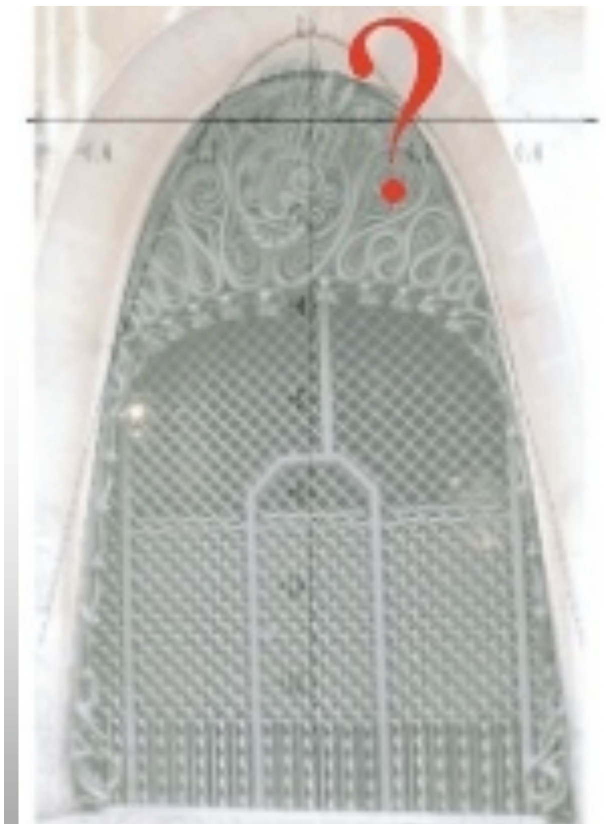
None of the "plausible" arches has fitted that gate with acceptable accuracy



Color code:

- catenary
- parabola
- circle
- conic

... but "others" maintain this arch is a catenary, at least, in some relaxed sense ...



Given the basis points P_0 and P_2 and the top point P_1, the computation of the corresponding catenary arch requires to solve an nonlinear equation with a main unknown, the scale factor a.

That is achieved in the application we have presented. But an stretched catenary has two unknowns, the quoted a and the scale factor k to apply to the height of a true catenary in order to stretch it. Related to this, we need to add the passing point P_P to our constrains.

Thus, we face now a system of two nonlinear equations. To avoid unnecessary work, we used a commercial mathematical software to do that.

Una vez calculadas las medidas en la fotografía insertada en AutoCAD, se realiza el cálculo de *a* y *b* (y, de aquí, *k*) imponiendo el paso por la base del arco (semiluz) y por un punto estimado de paso. El paso por la clave está garantizado a priori haciendo $h = flecha$ en la función **strcat**.

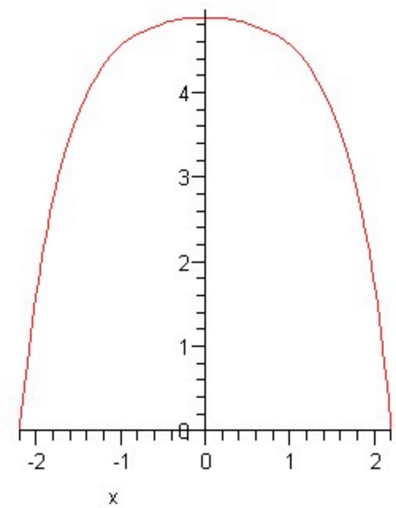
Datos obtenidos de la fotografía. (*deltx*,*delty*)=coordenadas de un punto de paso.

```
> luz:=4.4012:
  flecha:=4.8882:
  deltx:=1.8292:
  delty:=2.6492:
  sluz:=luz/2;
sluz := 2.200600000
```

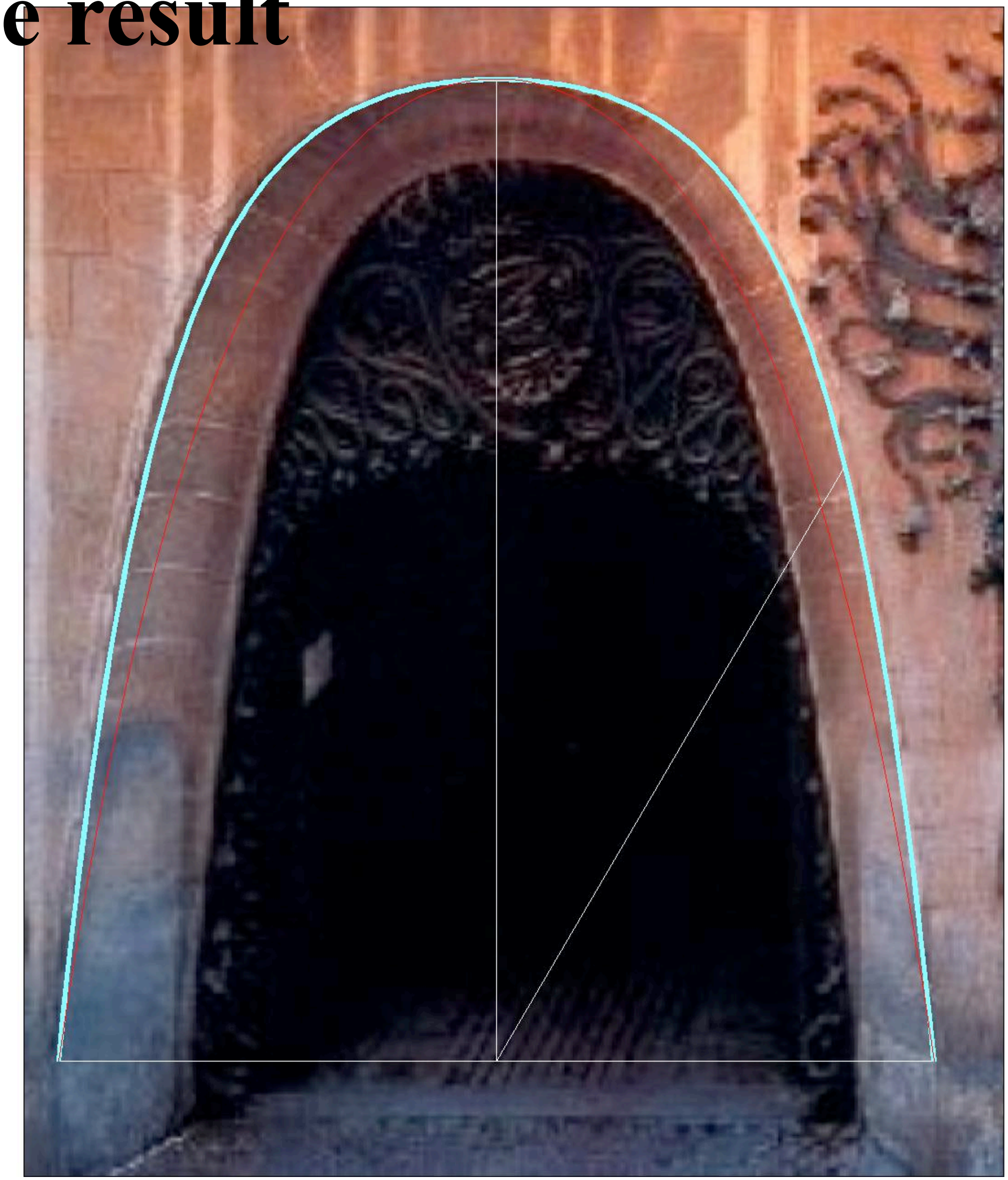
Cálculo de la solución, a partir de un sistema de ecuaciones no lineales y dos incógnitas (*a* y *b*).
El sistema se resuelve por métodos numéricos, tipo Newton-Raphson, que iteran a partir de una aproximación inicial; aquí se toma 0.5 y 0.15 como aproximaciones iniciales respecivas a los valores finales de *a* y *b*. La función Maple que realiza este tipo de cálculo es **fsolve**.

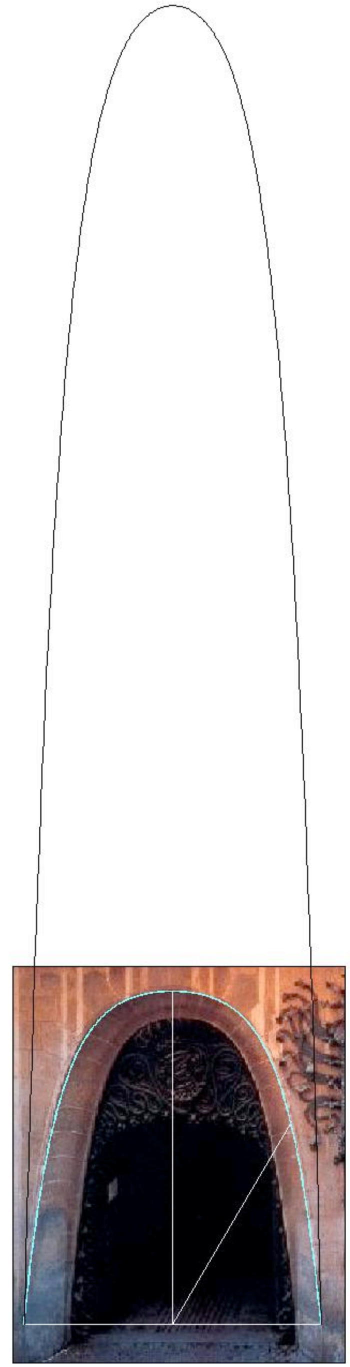
Dibujo de la solución.

```
> soluc:=fsolve({strcat(a,b,flecha)(sluz)=0.0,strcat(a,b,flecha)(deltx)=delty},{a=0.5,b=0.15});
soluc := {a = .4922124415, b = .1144227838}
> dib1:=plot(strcat(0.4922124415,0.1144227838,flecha)(x),x=-sluz..sluz,scaling=constrained):
  display(dib1);
```

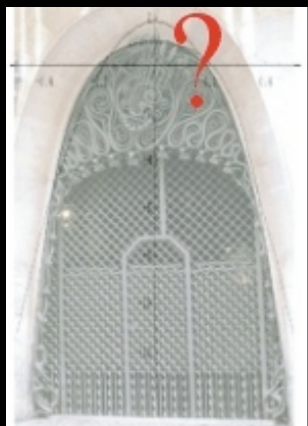


The result





- Reply for some of the “others”…



The hyperbolic cosine:
$y = 1.34 - 0.36\cosh(9.7x)$, fitted at 99.88%
but it did not go through the top:
for $x=0$, $y \approx 0.02$

We can easily give more importance to the fact that the curve should go through the top, by counting that point several times.

Counting the top 10 times: $y = 1.42 - 0.37\cosh(9.7x)$,
so that for $x=0$ the difference is but 0.01.

Counting the top 20 times: $y = 1.36 - 0.36\cosh(9.7x)$,
so that for $x=0$ the difference is ≈ 0.0.

- As for the problem of 'solving the non-linear equations', the answer lays can also be given by the use of … *polynomials!*

Actually, I did the curve fitting with an arbitrary polynomial of degree n:

$y = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots + b_nx^n$

It turned out it the answer was:

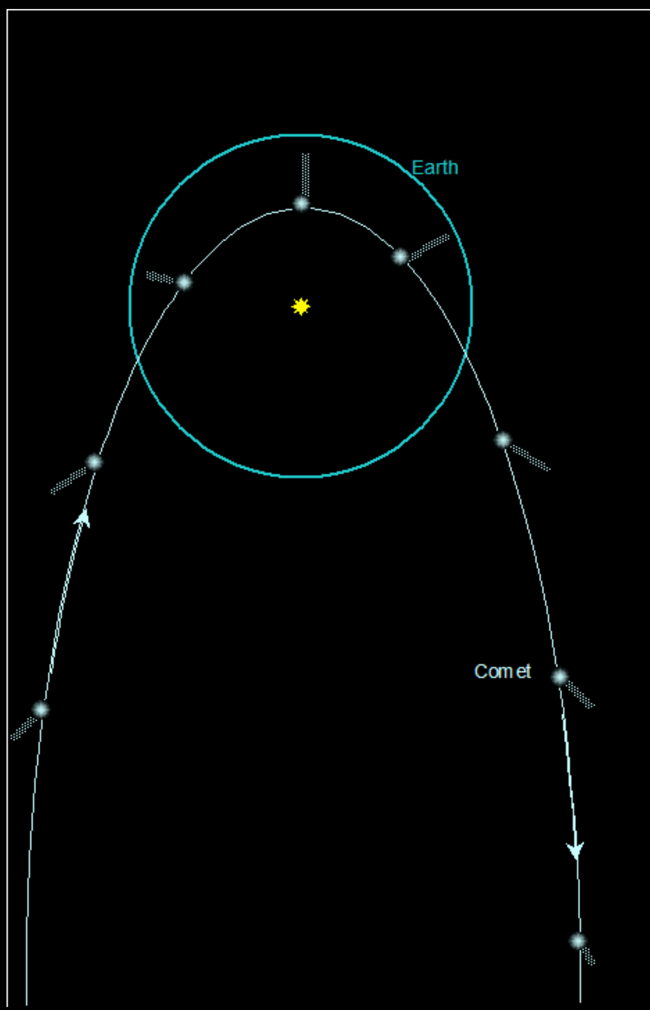
$y = 1.019 - 6.115\times10^{-16}x - 20.95x^2 - 1.10\times10^{-13}x^3 - 60.7x^4$
$- 9.9\times10^{-13}x^5 - 866.96x^6$

And this polynomial could be transformed into

$y \approx 1 - 21x^2 - 61x^4 - 867x^6$
$\approx 1 - (6.5x)^2/2! - (6.5x)^4/4! - (9.3x)^6/6!$

and this reminds the series of a cosh(ax)

- Why would there be a difference for curve fitting for, say …

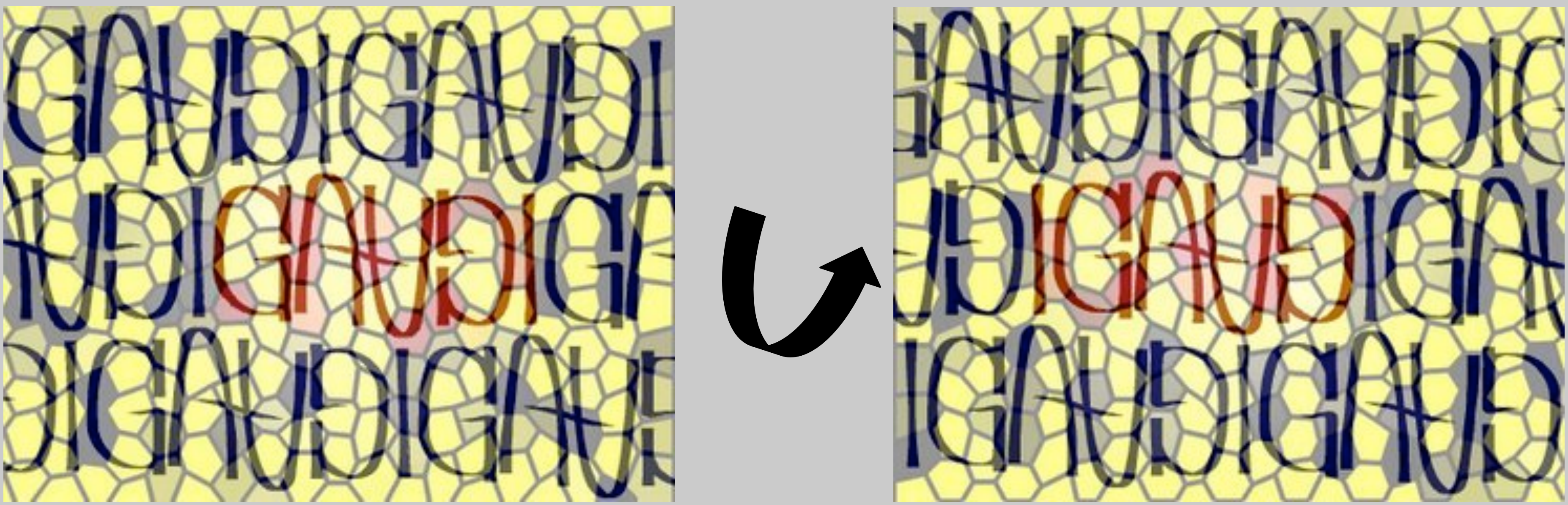


a comet…

…or an architectural curve?

And we could not agree on the end-joke either…

Should we observe it this way… *…or this way…*



Source: ambigrames.blogspot.com/